Last time: big picture on diagonalization of $B \in \mathbb{R}^{n \times n}$

$$B = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^{-1}$$

How to start from B and find P and $\lambda_1, \dots, \lambda_n$?

Step 1: construct characteristic polynomial of B:

$$\chi_B(t) = \det(t \cdot I_n - B)$$

Step 2: find roots $\lambda_1, \dots, \lambda_n$ of $\chi_B(t)$ (note they may be repeated)

$$\chi_B(t) = (t - \lambda_1) \dots (t - \lambda_n)$$

eigenvalues of B; may be complex

Step 3: $\forall \lambda_i$, find an eigenvector $v_i \neq 0$

$$\Updownarrow$$

$$B v_i = \lambda_i v_i$$

by finding a non-zero solution to $(B - \lambda_i \cdot I_n) v_i = 0$

Try to find as many linearly independent eigenvectors as possible

Diagonalization of B is possible $\Longleftrightarrow$ $\exists$ linearly independent eigenvectors v_1 for $\lambda_1, \ldots, v_n$ for λ_n

Indeed, we proved that with the choice $P = (v_1 \ldots\ldots v_n)$

we have
$$B = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^{-1}$$

Today: more details on eigenvalues and eigenvectors

DEF 19.1: the spectrum of a matrix B is the multiset of its eigenvalues (denoted $\mathrm{Spec}(B)$)

multiset: set with repetitions, e.g. $\{2,5,2\} = \{5,2,2\} = \{2,2,5\}$

How to see that the order in which we write the eigenvalues doesn't matter

if $B = P\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}P^{-1}$ then $B = P'\begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}P'^{-1}$

$\begin{pmatrix} 0 & 1 \\ \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ \end{pmatrix}$

Proof: let $P = P' \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ invertible, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ [cut off]

$$\Downarrow$$

$$P^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} P'^{-1}$$

$$B = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1} = P' \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\begin{pmatrix} 0 & \lambda_2 \\ \lambda_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{pmatrix}} P'^{-1}$$

$\square$

Examples of spectra

- $B = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix} \rightsquigarrow \mathrm{Spec}(B) = \{d_1, d_2, \dots, d_n\}$

- $B = \begin{pmatrix} d_1 & & & * \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$

  or

  $B = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ * & & & d_n \end{pmatrix}$

  $\rightsquigarrow \mathrm{Spec}(B) = \{d_1, d_2, \dots, d_n\}$

Proof: $\chi_B(t) = \det(t \cdot I_n - B) = \det \begin{pmatrix} t-d_1 & & & * \\ & t-d_2 & & \\ & & \ddots & \\ 0 & & & t-d_n \end{pmatrix}$ [cut off]

already factored
roots are $d_1, \dots, d_n$

$$= (t-d_1)(t-d_2)\dots(t-d_n)$$

$$\Rightarrow \operatorname{Spec}(B) = \{d_1, d_2, \dots, d_n\}.$$

- $D = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$, $D' = \begin{pmatrix} d_1' & & 0 \\ & \ddots & \\ 0 & & d_n' \end{pmatrix} \Rightarrow DD' = \begin{pmatrix} d_1 d_1' & & 0 \\ & \ddots & \\ 0 & & d_n d_n' \end{pmatrix}$

But $B = PDP^{-1}$, $B' = P'D'P'^{-1}$ $\rightsquigarrow$ in general, eigenvalues of BB' are not products of eig of B and eig of B'

Exception: if $P = P'$, $BB' = PD\underbrace{P^{-1}P}_{=I_n}D'P^{-1} = P(DD')P^{-1}$

- $\lambda \cdot I_n = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$ $\rightsquigarrow$ $\operatorname{Spec}(\lambda \cdot I_n) = \{\lambda, \dots, \lambda\}$

- $J_n^{(\lambda)} = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ Jordan block is NOT diagonalizable and $\operatorname{Spec}(J_n^{(\lambda)}) = \{\lambda, \dots, \lambda\}$

suppose $J_n^{(\lambda)}$ were diagonalizable $\Rightarrow J_n^{(\lambda)} = PDP^{-1}$

D would be the diagonal matrix of eigenvalues of $J_n^{(\lambda)}$

D would be the diagonal matrix ... $\{\lambda, \dots, \lambda\}$

$\Downarrow$

$$D = \lambda I_n \implies J_n^{(\lambda)} = PDP^{-1} = P \lambda I_n P^{-1} = \lambda P I_n P^{-1} = \lambda \cdot I_n$$

Contradiction.

Ex: $T : \mathbb{P}_d \longrightarrow \mathbb{P}_d$, T = derivative

basis $\{t^0, t^1, \dots, t^d\}$

$\updownarrow \quad \updownarrow \qquad \updownarrow$

$e_1 \quad e_2 \qquad e_{d+1}$

$f(t) = a_1 t^0 + a_2 t^1 + \dots + a_{d+1} t^d$

$$f = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{d+1} \end{pmatrix} \rightsquigarrow T(f) = \begin{pmatrix} a_2 \\ 2a_3 \\ \vdots \\ d a_{d+1} \\ 0 \end{pmatrix}$$

in this basis, $T(f) = B \cdot f$, $B = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & 0 & \ddots & \\ & & & 0 & d \\ & & & & 0 \end{pmatrix}$

not diagonalizable, because it's triangular with all 0's on the diagonal; if it were diagonalizable, then it would be $0 \cdot I_n$.

DEF 19.2: two square matrices A, B of same size are called *similar (conjugate)* if $B = PAP^{-1}$ for some invertible matrix P (denoted $A \sim B$)

THM 19.3: if A and B are similar, then

$$\mathrm{Spec}(A) = \mathrm{Spec}(B)$$

Caution: converse is not true, because $\begin{matrix} A = \lambda \cdot I_n \\ B = J_n^{(\lambda)} \end{matrix}$ have spectrum $\{\lambda, \dots, \lambda\}$ but they are not conjugate

Ex: $A = \begin{pmatrix} 2 & & 0 \\ & 2 & \\ 0 & & 5 \end{pmatrix} \sim \begin{pmatrix} 5 & & 0 \\ & 2 & \\ 0 & & 2 \end{pmatrix} = B$

because $B = PAP^{-1}$ where $P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$$\mathrm{Spec}(B) = \{\text{roots of } \chi_B(t)\}$$

real matrix

might be complex

real coefficients

THM 19.4: the complex roots of a real polynomial $f(t)$

always come in complex conjugate pairs, ...

$$f(\lambda)=0 \iff f(\bar{\lambda})=0$$

So if a matrix has $a+bi$ as an eigenvalue, it also has $a-bi$ as an eigenvalue.

E.g. $B=\begin{pmatrix} 2 & -4 \\ 5 & 3 \end{pmatrix}$ $\quad \chi_B(t)=\det(t\cdot I_2-B)=\det\begin{pmatrix} t-2 & +4 \\ -5 & t-3 \end{pmatrix}$

$$=(t-2)(t-3)+20=t^2-5t+26$$

$$\lambda_1=\frac{5}{2}+\frac{\sqrt{25-4\cdot 26}}{2}=\frac{5}{2}+\frac{\sqrt{-79}}{2}=\frac{5}{2}+\frac{\sqrt{79}}{2}\cdot i$$

complex conjugates

$$\lambda_2=\frac{5}{2}-\frac{\sqrt{25-4\cdot 26}}{2}=\frac{5}{2}-\frac{\sqrt{-79}}{2}=\frac{5}{2}-\frac{\sqrt{79}}{2}\cdot i$$

THM 19.4 continued: if v is an eigenvector of a real matrix B corresponding to $\lambda=a+bi$, then $\bar{v}$ is an eigenvector of B corresponding to $\bar{\lambda}=a-bi$

Proof: $Bv=\lambda v \Rightarrow \overline{Bv}=\overline{\lambda v} \Rightarrow \bar{B}\,\bar{v}=\bar{\lambda}\,\bar{v} \Rightarrow B\,\bar{v}=\bar{\lambda}\,\bar{v}$

($B=\bar{B}$ because real numbers are their own complex conjugates)

$(\bar{B} = B$ because real numbers

Finding eigenvectors in the above 2×2 example

$$B\, v_1 = \lambda_1 v_1 \iff (B - \lambda_1 \cdot I_2)\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\iff \begin{pmatrix} 2-\lambda_1 & -4 \\ 5 & 3-\lambda_1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

$$\iff \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{79}}{2} i & -4 \\ 5 & \frac{1}{2} - \frac{\sqrt{79}}{2} i \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0$$

An eigenvector should satisfy

$$\left(-\frac{1}{2} - \frac{\sqrt{79}}{2} i\right) x - 4y = 0 \iff y = \frac{1}{4}\left(-\frac{1}{2} - \frac{\sqrt{79}}{2} i\right) x$$

so $v_1 = \begin{pmatrix} 1 \\ \frac{1}{4}\left(-\frac{1}{2} - \frac{\sqrt{79}}{2} i\right) \end{pmatrix}$ should be an eigenvector as long as we can check

$$5 \cdot 1 + \left(\frac{1}{2} - \frac{\sqrt{79}}{2} i\right) \frac{1}{4}\left(-\frac{1}{2} - \frac{\sqrt{79}}{2} i\right) = 0$$

$$\Updownarrow$$

$$5 \cdot 1 + \frac{1}{4}\underbrace{\left(\frac{1}{2} - \frac{\sqrt{79}}{2} i\right)\left(-\frac{1}{2} - \frac{\sqrt{79}}{2} i\right)}_{-\frac{1}{4} + \cancel{\frac{\sqrt{79}}{4} i} - \cancel{\frac{\sqrt{79}}{4} i} + \frac{79}{4} \cdot i^2} = 0$$

-20

Summary: an eigenvector for $\lambda_1 = \frac{5}{2} + \frac{\sqrt{79}}{2} i$ is $v_1 = \begin{pmatrix} 1 \\ -\frac{1}{8} - \frac{\sqrt{79}}{8} i \end{pmatrix}$

$\Downarrow$

an eigenvector for $\lambda_2 = \frac{5}{2} - \frac{\sqrt{79}}{2} i$ is $v_2 = \begin{pmatrix} 1 \\ -\frac{1}{8} + \frac{\sqrt{79}}{8} i \end{pmatrix}$

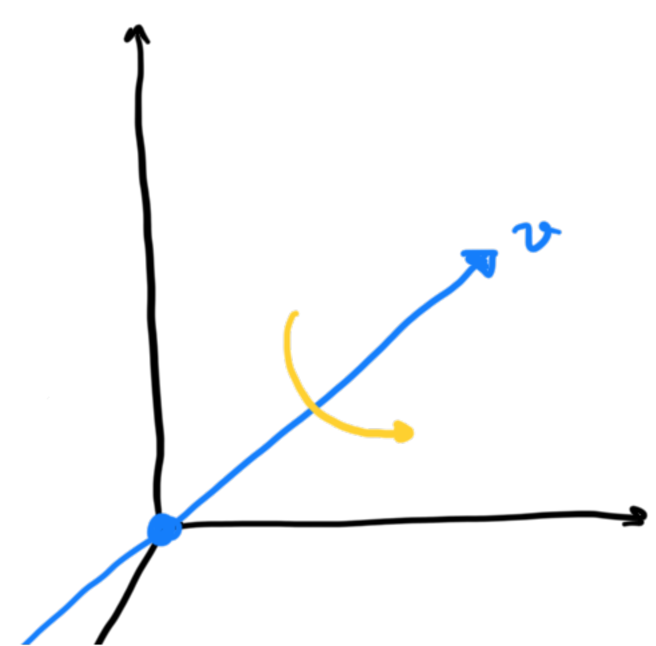
Sanity check: $B v_2 = \lambda_2 v_2$

$$\begin{pmatrix} 2 & -4 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{8} + \frac{\sqrt{79}}{8} i \end{pmatrix} = \begin{pmatrix} 2 + \frac{4}{8} - \frac{\sqrt{79}}{2} i \\ 5 - \frac{3}{8} + \frac{3\sqrt{79}}{8} i \end{pmatrix} = \begin{pmatrix} \frac{5}{2} - \frac{\sqrt{79}}{2} i \\ \frac{37}{8} + \frac{3\sqrt{79}}{8} i \end{pmatrix}$$

$$\left(\frac{5}{2} - \frac{\sqrt{79}}{2} i\right) \begin{pmatrix} 1 \\ -\frac{1}{8} + \frac{\sqrt{79}}{8} i \end{pmatrix} = \begin{pmatrix} \frac{5}{2} - \frac{\sqrt{79}}{2} i \\ \left(\frac{5}{2} - \frac{\sqrt{79}}{2} i\right)\left(-\frac{1}{8} + \frac{\sqrt{79}}{8} i\right) = -\frac{5}{16} + \frac{\sqrt{79}\, i}{16} + \frac{5\sqrt{79}}{16} i + \frac{79}{16} \end{pmatrix}$$

Question: why is every rotation in $\mathbb{R}^3$ around an axis?

It's because any real 3×3 matrix has at least one real eigenvector v



this is because $\chi_B(t)$ is a degree 3 = odd polynomial, and any odd degree real polynomial must have at least one real root λ

to real eigenvalues correspond real eigenvectors

$\Downarrow$

any odd sized matrix must have ≥ 1 real eigenvalue and eigenvector

DEF 19.5: assume that the distinct eigenvalues of a matrix $B \in \mathbb{R}^{n \times n}$ are $\lambda_1, \ldots, \lambda_r$ $(r \leq n)$

if $\chi_B(t) = (t-\lambda_1)^{d_1}(t-\lambda_2)^{d_2} \cdots (t-\lambda_r)^{d_r}$, then

$d_1, \ldots, d_r$ are called algebraic multiplicities of $\lambda_1, \ldots, \lambda_r$

Examples: $B = \begin{pmatrix} 2 & * & * \\ 0 & 3 & * \\ 0 & 0 & 5 \end{pmatrix} \Rightarrow \chi_B(t) = (t-2)(t-3)(t-5)$

has algebraic multiplicity 1

eigenvalue 2 has

eigenvalue 3 has algebraic multiplicity 1

eigenvalue 5 has algebraic multiplicity 1

$$B = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \Rightarrow \chi_B(t) = (t+4)^2(t-6)$$

eigenvalue -4 has algebraic multiplicity 2

eigenvalue 6 has algebraic multiplicity 1

$$B = \begin{pmatrix} -1 & 0 & 0 \\ * & -1 & 0 \\ * & * & -1 \end{pmatrix} \Rightarrow \chi_B(t) = (t+1)^3$$

eigenvalue -1 has algebraic multiplicity 3

THM 19.6: Sum of algebraic multiplicities $= n$

(also, if λ has alg. mult. d, then $\bar{\lambda}$ also has alg. mult. d)

DEF 19.7: say that λ is an eigenvalue of a matrix B; the geometric multiplicity of λ is the number

$$\dim \operatorname{Ker}(B - \lambda \cdot I_n) = \dim \{ v \in \mathbb{R}^n \mid Bv = \lambda v \}$$

$= \dim \{ \text{set of eigenvectors for } \lambda \} \cup \{0\}$

THM 19.8: $1 \leq \text{geom. mult of } \lambda \leq \text{alg. mult of } \lambda$

max # of linearly independent eigenvectors for λ

A matrix is diagonalizable $\iff$ all eigenvalues have geometric mult. = algebraic mult.